

COMPUTING THE CORE OF IDEALS IN ARBITRARY CHARACTERISTIC

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ABSTRACT. Let R be a local Gorenstein ring with infinite residue field of arbitrary characteristic. Let I be an R -ideal with $g = \text{ht } I > 0$, analytic spread ℓ , and let J be a minimal reduction of I . We further assume that I satisfies G_ℓ and $\text{depth } R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq \ell - g$. The question we are interested in is whether $\text{core}(I) = J^{n+1} : \sum_{b \in I} (J, b)^n$ for $n \gg 0$. In the case of analytic spread one Polini and Ulrich show that this is true with even weaker assumptions ([15, Theorem 3.4]). We give a negative answer to this question for higher analytic spreads and suggest a formula for the core of such ideals.

1. INTRODUCTION

Throughout let R be a Noetherian ring. If R is a Noetherian local ring with maximal ideal \mathfrak{m} then we denote the residue field of R by $k = R/\mathfrak{m}$. Let I be an R -ideal. In order to study an ideal I , Northcott and Rees introduced the notion of a reduction of an ideal. A reduction in general is a simplification of the ideal itself. Recall that a *reduction* of an ideal I is a subideal J such that $I^{n+1} = JI^n$, for some nonnegative integer n ([14]). This condition is equivalent to I being integral over J . Moreover, reductions preserve a number of properties of the ideal and thus it is customary to shift the attention from the ideal to its reductions. In the case that R is a Noetherian local ring we may consider *minimal reductions*, which are minimal with respect to inclusion. Northcott and Rees prove that if the residue field k of R is infinite then minimal reductions do indeed exist and they correspond to Noether normalizations of the special fiber ring $\mathcal{F}(I) := \bigoplus_{i \geq 0} I^i/\mathfrak{m}I^i = R/\mathfrak{m} \oplus I/\mathfrak{m}I \oplus \dots \oplus I^i/\mathfrak{m}I^i \oplus \dots$ of I ([14]). In particular this shows that minimal reductions are not unique. Recall that the *analytic spread* of I , $\ell(I)$, is the Krull dimension of the special fiber ring $\mathcal{F}(I)$, i.e., $\ell = \ell(I) = \dim \mathcal{F}(I)$. If k is infinite Northcott and Rees also show that for any minimal reduction J of I one has $\mu(J) = \ell(I)$, where $\mu(J)$ denotes the minimal number of generators of J ([14]).

In order to counteract the lack of uniqueness of minimal reductions Rees and Sally consider the intersection over all (minimal) reductions, namely the *core* of the ideal ([17]). Then $\text{core}(I) = \bigcap_J J$, where J is a (minimal) reduction of I . The core arises naturally in the context of Briançon–Skoda kind of theorems. If R is a regular local ring of dimension d and I is an R -ideal, then the Briançon–Skoda theorem states that $\overline{I^d} \subset J$, for every reduction J of I , or equivalently $\overline{I^d} \subset \text{core}(I)$, where ‘ $\overline{}$ ’ denotes the integral closure of the corresponding ideal. Huneke and Swanson ([8]) showed a connection between the work of Lipman ([12]) on the adjoint of an ideal and the core. The core is a priori an infinite intersection. Hence there is significant difficulty in computing this ideal. The question of finding explicit formulas that compute the core has been addressed in the work of Corso,

Huneke, Hyry, Polini, Smith, Swanson, Trung, Ulrich and Vitulli ([2], [3], [8], [9], [10], [11], [15], [16]). Moreover, Hyry and Smith have discovered a connection with a conjecture by Kawamata on the non-vanishing of sections of line bundles ([11]).

In this paper we are primarily interested in a formula for the core of an ideal shown by Polini and Ulrich which states:

Theorem 1.1. ([15, Theorem 4.5]) *Let R be a local Gorenstein ring with infinite residue field k , let I be an R -ideal with $g = \text{ht } I > 0$ and $\ell = \ell(I)$, and let J be a minimal reduction of I with reduction number r . Assume I satisfies G_ℓ and $\text{depth } R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq \ell - g$, and either $\text{char } k = 0$ or $\text{char } k > r - \ell + g$. Then*

$$\text{core}(I) = J^{n+1} : I^n$$

for every $n \geq \max\{r - \ell + g, 0\}$.

The goal will be clear once it is understood how the formula in Theorem 1.1 arises. In general Polini and Ulrich show that:

Theorem 1.2. ([15, Remark 4.8]) *Let R be a local Gorenstein ring with infinite residue field, let I be an R -ideal with $g = \text{ht } I > 0$ and $\ell = \ell(I)$, and let J be a minimal reduction of I with reduction number r . Assume I satisfies G_ℓ and $\text{depth } R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq \ell - g$. Then*

$$(1.1) \quad J^{n+1} : I^n \subset \text{core}(I) \subset J^{n+1} : \sum_{b \in I} (J, b)^n$$

for every $n \geq \max\{r - \ell + g, 0\}$.

These inclusions hold in any characteristic. The condition on the characteristic of the residue field in Theorem 1.1 implies that the two bounds for the core in equation (1.1) coincide. This gives the formula in Theorem 1.1.

When the analytic spread of I is one, Polini and Ulrich also show the following:

Theorem 1.3. ([15, Theorem 3.4]) *Let R be a local Cohen–Macaulay ring with infinite residue field, let I be an R -ideal with $\ell(I) = \text{ht } I = 1$, and let J be a minimal reduction of I . Then for $n \gg 0$*

$$\text{core}(I) = J^{n+1} : \sum_{b \in I} (J, b)^n.$$

Notice that Theorem 1.3 holds in any characteristic. In the same paper Polini and Ulrich also exhibit a class of examples where $\ell(I) = 1$ and $\text{core}(I) \neq J^{n+1} : I^n$ ([15, Example 4.9]). Thus a natural question arises:

Question 1.4. Under the same assumptions as in Theorem 1.1, except for the condition on the characteristic of k , is $\text{core}(I) = J^{n+1} : \sum_{b \in I} (J, b)^n$ for some $n \gg 0$?

The purpose of this paper is to answer Question 1.4. In order to answer this question we first seek to better understand the ideal $J^{n+1} : \sum_{b \in I} (J, b)^n$. One of the difficulties lies in computing $\sum_{b \in I} (J, b)^n$.

We devote Section 2 to understanding this ideal. In Theorem 2.4 we give an explicit algorithm for computing this ideal. Once we are able to compute it we are interested in the behaviour of the ideal $J^{n+1} : \sum_{b \in I} (J, b)^n$, which we address in Section 3. In Section 4 we finally answer Question 1.4.

Before we proceed any further we need to explain some of the conditions that are used in Theorem 1.1 and throughout this paper. Let R be a Noetherian ring, I an R -ideal and s an integer. We say that I satisfies G_s if $\mu(I_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}}$ for every $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} \leq s - 1$. If R is a Noetherian local ring of dimension d and \mathfrak{m} is the maximal ideal of R , then any \mathfrak{m} -primary ideal satisfies G_d . An additional technical condition that is connected with the study of the core is the assumption $\text{depth } R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq \ell - g$, where g is the height of I and ℓ is the analytic spread of I .

Let R be a local Gorenstein ring with maximal ideal \mathfrak{m} and infinite residue field and let I be an R -ideal with height g and analytic spread ℓ . Then I satisfies G_{ℓ} and $\text{depth } R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq \ell - g$ in the following cases:

- (a) I is \mathfrak{m} -primary, or more generally I is equimultiple which means $\ell = g$.
- (b) I is a one-dimensional generic complete intersection ideal, or more generally I is a generic complete intersection Cohen–Macaulay ideal with $\ell \leq g + 1$ ([1, p. 259]).

In the presence of the G_{ℓ} property the depth condition on the powers of I as above is satisfied if I is perfect of height 2, or if I is perfect Gorenstein of height 3, or more generally if I is in the linkage class of a complete intersection ideal (licci) ([7, 1.11]).

An interesting invariant of an ideal I is the reduction number of I . The *reduction number of I with respect to J* is the integer $r_J(I) = \min \{ n \mid I^{n+1} = JI^n \}$, where J is a reduction of I . The *reduction number of I* , $r(I)$, is defined to be $\min \{ r_J(I) \mid J \text{ minimal reduction of } I \}$. The reduction number of I is connected with the study of blowup algebras and their Cohen–Macaulayness.

2. THE IDEAL K_n

Let R be a Noetherian local ring with infinite residue field and let I be an R -ideal. Let J be a minimal reduction of I . As a starting point of our work we first seek to better understand the ideal $\sum_{b \in I} (J, b)^n$ as it is connected with the core of I by work of Polini and Ulrich ([15]). Our first goal is to find an efficient way to compute this ideal. We start investigating such an ideal in a general setting.

Definition 2.1. Let R be a Noetherian ring and let $J \subset I$ be R -ideals. Let n be a positive integer. We denote by $K_n(J, I)$ the R -ideal $\sum_{b \in I} (J, b)^n$ and by $L_n(J, I)$ the R -ideal $J^{n+1} : K_n(J, I)$. When the ideal I is understood we will denote these ideals by $K_n(J)$ and $L_n(J)$, respectively. If in addition the ideal J is understood then we will use K_n and L_n , respectively.

The following lemma gives an explicit description of a (not necessarily minimal) generating set for $K_n(J, I)$.

Lemma 2.2. *Let R be a Noetherian local ring with infinite residue field. Let T be an algebra over R and let $J \subset I$ be T -ideals. Assume that $J = (f_1, \dots, f_t)$ and $I = (J, f_{t+1}, \dots, f_m)$. Then*

$$K_n(J, I) = (\{ \prod_{j=t+1}^{m-1} \binom{n - \sum_{i=1}^{j-1} v_i}{v_j} f_1^{v_1} \cdots f_m^{v_m} \}),$$

where v_1, v_2, \dots, v_m range over all nonnegative integers with $v_1 + v_2 + \dots + v_m = n$.

Proof. Let k denote the residue field of R . Let A be the ideal $(\{ \prod_{j=t+1}^{m-1} \binom{n - \sum_{i=1}^{j-1} v_i}{v_j} f_1^{v_1} \cdots f_m^{v_m} \})$, where $v_1 + v_2 + \dots + v_m = n$.

We first show that $K_n = K_n(J, I) \subset A$. It suffices to check that the generators of $K_n(J, I)$ are in A . Let f be such a generator. We may assume that f is of the form $f_1^{v_1} \cdots f_t^{v_t} y^{n-s}$, where $v_1 + \dots + v_t = s$, $0 \leq s \leq n$, and $y = \sum_{i=t+1}^m f_i g_i$, with $g_i \in T$ for $t+1 \leq i \leq m$. Then

$$\begin{aligned} f &= f_1^{v_1} \cdots f_t^{v_t} (f_{t+1} g_{t+1} + \cdots + f_m g_m)^{n-s} \\ &= \sum_{\underline{v}} \beta_{\underline{v}} f_1^{v_1} \cdots f_t^{v_t} f_{t+1}^{v_{t+1}} \cdots f_m^{v_m} g_{t+1}^{v_{t+1}} \cdots g_m^{v_m}, \end{aligned}$$

where

- (a) $\underline{v} = (v_{t+1}, \dots, v_m)$;
- (b) $v_{t+1} + \dots + v_m = n - s$ and $v_1 + \dots + v_t = s$, and
- (c) $\beta_{\underline{v}} = \prod_{j=t+1}^{m-1} \binom{n - \sum_{i=1}^{j-1} v_i}{v_j}$.

Hence clearly $f \in A$ and $K_n \subset A$. It remains to show that $A \subset K_n$.

Fix v_1, \dots, v_t . Let $N = n - \sum_{i=1}^t v_i$. As above write $\beta_{\underline{v}} = \prod_{j=t+1}^{m-1} \binom{n - \sum_{i=1}^{j-1} v_i}{v_j}$, where $\underline{v} = (v_{t+1}, \dots, v_m)$ and $v_{t+1} + \dots + v_m = N$. We prove that

$$\beta_{\underline{v}} f_1^{v_1} \cdots f_m^{v_m} \in K_n(J).$$

Let M_1, \dots, M_D be the monomials of degree N in the variables X_{t+1}, \dots, X_m . Then the elements $\beta_{\underline{v}} f_1^{v_1} \cdots f_m^{v_m}$ become $\beta_i f_1^{v_1} \cdots f_t^{v_t} M_i(\underline{f})$, where $\underline{f} = (f_{t+1}, \dots, f_m)$ and $1 \leq i \leq D$. We choose $\alpha_{t+1}, \dots, \alpha_m$ elements in R such that for all $1 \leq i \leq D$ the images of $M_i(\underline{\alpha})$ in k are distinct, where $\underline{\alpha} = (\alpha_{t+1}, \dots, \alpha_m)$. Since the residue field k is infinite it is possible to choose such elements. Then for $0 \leq e \leq D-1$ we have

$$(2.1) \quad K_n \ni f_1^{v_1} \cdots f_t^{v_t} \left(\sum_{j=t+1}^m \alpha_{f_j}^e \right)^N = f_1^{v_1} \cdots f_t^{v_t} \sum_{i=1}^D \beta_i (M_i(\underline{\alpha}))^e M_i(\underline{f}).$$

Let B denote the $D \times D$ matrix whose (j, i) entry is $(M_i(\underline{\alpha}))^{j-1}$ and let C denote the $D \times 1$ matrix whose i^{th} entry $\beta_i f_1^{v_1} \cdots f_t^{v_t}$ is $M_i(\underline{f})$. The entries of BC are in K_n according to equation (2.1). Notice that B is a Vandermondt matrix and hence the determinant of it is the product of the differences of all $M_i(\underline{\alpha})$. By the choice of $\underline{\alpha}$ these differences have non-zero images in k , and therefore are units

in R and thus in T . This implies that $\det B$ is a unit and hence B is invertible. Therefore the entries of C are in K_n . \square

Definition 2.3. Let R be a Noetherian local ring with infinite residue field k . Let $I = (f_1, \dots, f_m)$ be an R -ideal and let t be a fixed positive integer. We say that b_1, \dots, b_t are t general elements in I if there exists a dense open subset U of \mathbb{A}_k^{tm} such that for $1 \leq i \leq t$ and $1 \leq j \leq m$ we have that $b_i = \sum_{j=1}^m \lambda_{ij} f_j$, where $\underline{\lambda} = [\lambda_{ij}]_{ij} \in \mathbb{A}_R^{tm}$ and $\underline{\lambda} \in U$ vary in U , where $\underline{\lambda}$ is the image of $\underline{\lambda}$ in \mathbb{A}_k^{tm} .

The ideal J is called a *general minimal reduction* of I if J is a reduction of I generated by $\ell(I)$ general elements in I .

Theorem 2.4. Let R be a Noetherian local ring with infinite residue field. Let $J \subset I$ be R -ideals. Let n be a positive integer and $K_n = \sum_{b \in I} (J, b)^n$. For a fixed integer t let b_1, \dots, b_{t+1} be $t+1$ general elements in I . Set $C_j = \sum_{i=1}^j (J, b_i)^n$ for $1 \leq j \leq t+1$. Assume that $C_t = C_{t+1}$. Then

$$K_n = \sum_{i=1}^t (J, b_i)^n = C_t.$$

Proof. Assume that $J = (f_1, \dots, f_s)$ and $I = (f_1, \dots, f_s, f_{s+1}, \dots, f_m)$. Let k denote the residue field of R . Clearly $C_t = \sum_{i=1}^t (J, b_i)^n \subset K_n$. Notice that there exists a positive integer $t' > t$ such that

$$K_n = \sum_{b \in I} (J, b)^n = \sum_{i=1}^{t'} (J, y_i)^n \text{ for some } y_i \in I.$$

We consider the natural projection maps $\pi_j : \mathbb{A}_k^{t'm} \rightarrow \mathbb{A}_k^{(t+1)m}$ where:

- (a) $t+1 \leq j \leq t'$;
- (b) $\pi_j((\underline{a}_1, \dots, \underline{a}_t, \underline{a}_{t+1}, \dots, \underline{a}_j, \dots, \underline{a}_{t'})) = (\underline{a}_1, \dots, \underline{a}_t, \underline{a}_j)$, and
- (c) $\underline{a}_i \in \mathbb{A}_k^m$.

Let $\underline{\lambda} = [\lambda_{ij}]_{i,j} \in \mathbb{A}_R^{(t+1)m}$ and let $\underline{\lambda}$ denote the image of $\underline{\lambda}$ in $\mathbb{A}_k^{(t+1)m}$. By our assumption there exists a dense open subset $U \subset \mathbb{A}_k^{(t+1)m}$ such that $C_{t+1} = \sum_{i=1}^{t+1} (J, b_i)^n = \sum_{i=1}^t (J, b_i)^n = C_t$, where $b_i = \sum_{j=1}^m \lambda_{ij} f_j$ for $1 \leq i \leq t+1$ and $\underline{\lambda} \in U$. Notice that $V = \bigcap_{j=t+1}^{t'} \pi_j^{-1}(U)$ is a dense open subset in $\mathbb{A}_k^{t'm}$.

For $1 \leq i \leq t'$ let $b_i = \sum_{j=1}^m \lambda_{ij} f_j$, where $\underline{\lambda} \in V$. By the construction of V one has that $C_{t'} = C_t$. So it suffices to show that $K_n = C_{t'}$.

Let $T = R[X_{ij}]$, where $1 \leq i \leq t'$ and $1 \leq j \leq m$. For $1 \leq i \leq t'$ let $y_i = \sum_{j=1}^m \lambda_{ij}^0 f_j$ with $\lambda_{ij}^0 \in \mathbb{R}$, and write $\underline{\lambda}^0 = [\lambda_{ij}^0]_{i,j}$ and $\underline{X} = [X_{ij}]_{i,j}$. Consider the T -ideal $\widetilde{K}_n = \sum_{i=1}^{t'} (JT, \sum_{j=1}^m X_{ij} f_j)^n$ and the R -homomorphisms $\pi_{\underline{\lambda}} : T \rightarrow R$ that send \underline{X} to $\underline{\lambda}$, where $\underline{\lambda} \in \mathbb{A}_R^{t'm}$. Notice that $\pi_{\underline{\lambda}^0}(\widetilde{K}_n) = K_n$. Therefore we have

$$(2.2) \quad K_n T \subset \widetilde{K}_n + (\underline{X} - \underline{\lambda}^0),$$

and

$$(2.3) \quad \widetilde{K}_n \subset K_n(JT, IT) = K_n T,$$

where the last equality holds by Lemma 2.2.

Write $\mathfrak{m}_{\underline{\lambda}}$ for the maximal ideals $(\mathfrak{m}, \underline{X} - \underline{\lambda})$ of T . Localizing equation (2.3) at these maximal ideals gives

$$(\widetilde{K}_n)_{\mathfrak{m}_{\underline{\lambda}}} \subset K_n T_{\mathfrak{m}_{\underline{\lambda}}},$$

and combining this with equation (2.2) yields

$$(2.4) \quad K_n T_{\mathfrak{m}_{\underline{\lambda}^0}} = (\widetilde{K}_n)_{\mathfrak{m}_{\underline{\lambda}^0}} + (\underline{X} - \underline{\lambda}^0) \cap K_n T_{\mathfrak{m}_{\underline{\lambda}^0}}.$$

Since $\underline{X} - \underline{\lambda}^0$ is a regular sequence on $T_{\mathfrak{m}_{\underline{\lambda}^0}}/K_n T_{\mathfrak{m}_{\underline{\lambda}^0}}$, equation (2.4) becomes

$$K_n T_{\mathfrak{m}_{\underline{\lambda}^0}} = (\widetilde{K}_n)_{\mathfrak{m}_{\underline{\lambda}^0}} + (\underline{X} - \underline{\lambda}^0) K_n T_{\mathfrak{m}_{\underline{\lambda}^0}}$$

and thus

$$(2.5) \quad K_n T_{\mathfrak{m}_{\underline{\lambda}^0}} = (\widetilde{K}_n)_{\mathfrak{m}_{\underline{\lambda}^0}}$$

by Nakayama's lemma.

Equation (2.5) allows us to conclude that $M_{\mathfrak{m}_{\underline{\lambda}^0}} = 0$ for the T -module $M = \frac{\widetilde{K}_n + K_n T}{\widetilde{K}_n}$. Hence $\mathfrak{m}_{\underline{\lambda}^0} \notin \text{Supp}(M)$ and thus there exists a dense open set $U \subset \mathbb{A}_k^{mt'}$ such that $M_{\mathfrak{m}_{\underline{\lambda}}} = 0$ for all $\underline{\lambda} \in U$, where $\underline{\lambda} \in \mathbb{A}_R^{mt'}$ and $\underline{\lambda}$ denotes the image of $\underline{\lambda}$ in $\mathbb{A}_k^{mt'}$. Therefore $\frac{(\widetilde{K}_n)_{\mathfrak{m}_{\underline{\lambda}}} + K_n T_{\mathfrak{m}_{\underline{\lambda}}}}{(\widetilde{K}_n)_{\mathfrak{m}_{\underline{\lambda}}}} = 0$ for all $\underline{\lambda} \in U$. In other words, for all $\underline{\lambda} \in U$ we have $(\widetilde{K}_n)_{\mathfrak{m}_{\underline{\lambda}}} + K_n T_{\mathfrak{m}_{\underline{\lambda}}} = (\widetilde{K}_n)_{\mathfrak{m}_{\underline{\lambda}}}$. Let $\rho_{\underline{\lambda}} : T_{\mathfrak{m}_{\underline{\lambda}}} \rightarrow R$ be the R -homomorphism that sends \underline{X} to $\underline{\lambda}$. Then the above equation yields $\rho_{\underline{\lambda}}((\widetilde{K}_n)_{\mathfrak{m}_{\underline{\lambda}}} + K_n T_{\mathfrak{m}_{\underline{\lambda}}}) = \rho_{\underline{\lambda}}((\widetilde{K}_n)_{\mathfrak{m}_{\underline{\lambda}}})$ for all $\underline{\lambda} \in U$. Hence $\pi_{\underline{\lambda}}(\widetilde{K}_n) + K_n = \pi_{\underline{\lambda}}(\widetilde{K}_n)$ for all $\underline{\lambda} \in U$. As $\pi_{\underline{\lambda}}(\widetilde{K}_n) \subset K_n$ we conclude that $K_n = \pi_{\underline{\lambda}}(\widetilde{K}_n) = C_{t'}$ for all $\underline{\lambda}$ in a suitable dense open subset of $\mathbb{A}_k^{t'm}$. \square

Remark 2.5. Notice that Theorem 2.4 provides an algorithm for computing the ideal K_n for any positive integer n . We apply this algorithm in computations using the computer algebra program Macaulay 2 ([4]) in Section 4.

3. THE IDEAL L_n

In light of the algorithm given in Theorem 2.4 we are now able to compute the ideals $L_n = L_n(J) = J^{n+1} : \sum_{b \in I} (J, b)^n$. Recall that our goal is to determine whether $\text{core}(I) = L_n$ for $n \gg 0$ (Question 1.4).

However, determining what $n \gg 0$ means is another challenge of its own. If $\text{core}(I) \neq L_n$ for some n then in principle there is still a possibility that $\text{core}(I) = L_m$ for $m > n$. We need to determine how one can effectively decide when $\text{core}(I) \neq L_n$ for all $n > 0$. In this section we will prove that the ideals L_n stabilize past a computable integer (Theorem 3.4). This integer is related to the reduction number of a certain ideal. We begin our exploration by determining the reduction numbers of the ideals (J, b) , where b is a general element in I and J is a reduction of I .

Definition 3.1. Let R be a Noetherian local ring. Let I be an R -ideal and J a reduction of I . Assume $I = (f_1, \dots, f_m)$ and write $T = R[X_1, \dots, X_m]$, where X_1, \dots, X_m are variables over R . Let $\tilde{K} = (J, \sum_{j=1}^m X_j f_j)$. We define the integer s to be $r_{JT_m T}(\tilde{K}_{mT})$.

Notice that since J is a reduction of I it follows that JT is a reduction of IT . Hence $a = \sum_{j=1}^m X_j f_j \in IT$ is integral over JT and thus JT is a reduction of \tilde{K} .

Lemma 3.2. Let R be a Noetherian local ring with infinite residue field. Let I be an R -ideal and J a reduction of I . Let s be the integer as in Definition 3.1. If b is a general element in I , then $r_J((J, b)) \leq s$.

Proof.

Let k denote the residue field of R . Let $\tilde{M} = \tilde{K}^{s+1}/J\tilde{K}^s$, $\underline{X} = [X_1, \dots, X_m]$, and $\underline{\lambda} = [\lambda_1, \dots, \lambda_m] \in \mathbb{A}_R^m$. Write $\mathfrak{m}_{\underline{\lambda}}$ for the maximal ideals $(\mathfrak{m}, \underline{X} - \underline{\lambda})$ of T and consider the R -homomorphisms $\pi_{\underline{\lambda}} : T \rightarrow R$ that send \underline{X} to $\underline{\lambda}$.

From the choice of s we have that $\tilde{M}_{\mathfrak{m}T} = 0$ and hence $\mathfrak{m}T \notin \text{Supp}(\tilde{M})$. Thus there exists a dense open subset $U \subset \mathbb{A}_k^m$ such that $\tilde{M}_{\mathfrak{m}_{\underline{\lambda}}} = 0$ for every $\underline{\lambda} \in U$, where $\underline{\lambda}$ denotes the image of $\underline{\lambda}$ in \mathbb{A}_k^m . Therefore for all $\underline{\lambda} \in U$

$$(3.1) \quad \tilde{K}_{\mathfrak{m}_{\underline{\lambda}}}^{s+1} = (J\tilde{K}^s)_{\mathfrak{m}_{\underline{\lambda}}}.$$

In addition we consider the evaluation maps $\rho_{\underline{\lambda}} : T_{\mathfrak{m}_{\underline{\lambda}}} \rightarrow R$ that send \underline{X} to $\underline{\lambda}$. Then for every $\underline{\lambda} \in U$ we have $\rho_{\underline{\lambda}}(\tilde{K}_{\mathfrak{m}_{\underline{\lambda}}}^{s+1}) = \rho_{\underline{\lambda}}((J\tilde{K}^s)_{\mathfrak{m}_{\underline{\lambda}}})$ according to equation (3.1). In other words $(J, b)^{s+1} = J(J, b)^s$, whenever $b = \sum_{j=1}^m \lambda_j f_j$, $\underline{\lambda} = [\lambda_1, \dots, \lambda_m]$, and $\underline{\lambda} \in U$. Thus $r_J((J, b)) \leq s$. \square

The integer s is in general difficult to compute. However if the ideal I is \mathfrak{m} -primary then the following proposition gives a way to compute this integer.

Proposition 3.3. *Let R be a Noetherian local ring that is an epimorphic image of a Cohen–Macaulay ring. Let \mathfrak{m} be the maximal ideal of R and assume that $k = R/\mathfrak{m}$ is infinite. Let I be an \mathfrak{m} –primary ideal and J a reduction of I . Then $r_J((J, b)) = s$, where b is a general element in I .*

Proof. According to Lemma 3.2 we have that $JT_{\mathfrak{m}T}$ is a reduction of $\tilde{K}_{\mathfrak{m}T}$ and $r_J((J, b)) \leq s$. Following the notation in the proof of Lemma 3.2 we write $\mathfrak{m}_{\underline{\lambda}}$ for the maximal ideals $(\mathfrak{m}, \underline{X} - \underline{\lambda})$ of T , $\underline{X} = [X_1, \dots, X_m]$, and $\underline{\lambda} = [\lambda_1, \dots, \lambda_m] \in \mathbb{A}_R^m$. Let k denote the residue field of R .

As $J^s \subset \tilde{K}^s$ and J is \mathfrak{m} –primary we conclude that $(T/\tilde{K}^s)_{(\mathfrak{m}T)}$ is Artinian and thus Cohen–Macaulay. By the openness of the Cohen–Macaulay locus [13, Theorem 24.5] there exists a dense open subset U in \mathbb{A}_k^m such that $(T/\tilde{K}^s)_{\mathfrak{m}_{\underline{\lambda}}}$ is Cohen–Macaulay for all $\underline{\lambda} \in U$, where $\underline{\lambda}$ denotes the image of $\underline{\lambda}$ in \mathbb{A}_k^m . Write $b_{\underline{\lambda}} = \sum_{j=1}^m \lambda_j f_j$ for $\underline{\lambda} \in U$ and let $W = \{b \mid b = b_{\underline{\lambda}} \text{ for some } \underline{\lambda} \in \mathbb{A}_R^m \text{ such that } \underline{\lambda} \in U\}$.

Suppose that $(J, b)^s = J(J, b)^{s-1}$ for some $b \in W$. Notice that $\sqrt{\tilde{K}^s} = \mathfrak{m}T$ and thus $\dim(T/\tilde{K}^s)_{\mathfrak{m}_{\underline{\lambda}}} = m$. Note that $\frac{(T/\tilde{K}^s)_{\mathfrak{m}_{\underline{\lambda}}}}{(\underline{X} - \underline{\lambda})} \simeq R/(J, b)^s$ is an Artinian ring since I is \mathfrak{m} –primary. In addition $(T/\tilde{K}^s)_{\mathfrak{m}_{\underline{\lambda}}}$ is a Cohen–Macaulay ring of dimension m and the sequence $\underline{X} - \underline{\lambda}$ consists of m elements in $\mathfrak{m}_{\underline{\lambda}}$. Hence $\underline{X} - \underline{\lambda}$ is a regular sequence on $(T/\tilde{K}^s)_{\mathfrak{m}_{\underline{\lambda}}}$. Consider the R –homomorphisms $\rho_{\underline{\lambda}} : T_{\mathfrak{m}_{\underline{\lambda}}} \rightarrow R$ that send \underline{X} to $\underline{\lambda}$. Then since $(J, b)^s = J(J, b)^{s-1}$ we have $\rho_{\underline{\lambda}}(\tilde{K}_{\mathfrak{m}_{\underline{\lambda}}}^s) = \rho_{\underline{\lambda}}((J\tilde{K}^{s-1})_{\mathfrak{m}_{\underline{\lambda}}})$. Thus

$$\begin{aligned} \tilde{K}_{\mathfrak{m}_{\underline{\lambda}}}^s &= (J\tilde{K}^{s-1})_{\mathfrak{m}_{\underline{\lambda}}} + (\underline{X} - \underline{\lambda}) \cap \tilde{K}_{\mathfrak{m}_{\underline{\lambda}}}^s \\ &= (J\tilde{K}^{s-1})_{\mathfrak{m}_{\underline{\lambda}}} + (\underline{X} - \underline{\lambda})\tilde{K}_{\mathfrak{m}_{\underline{\lambda}}}^s, \end{aligned}$$

where the last equality holds since $\underline{X} - \underline{\lambda}$ is a regular sequence on $(T/\tilde{K}^s)_{\mathfrak{m}_{\underline{\lambda}}}$.

Finally by Nakayama’s lemma $\tilde{K}_{\mathfrak{m}_{\underline{\lambda}}}^s = (J\tilde{K}^{s-1})_{\mathfrak{m}_{\underline{\lambda}}}$ and therefore $\tilde{K}_{\mathfrak{m}T}^s = (J\tilde{K}^{s-1})_{\mathfrak{m}T}$, which is a contradiction. \square

The following theorem makes an effective use of the integer s as in Definition 3.1.

Theorem 3.4. *Let R be a local Gorenstein ring with infinite residue field k . Let I be an R –ideal with $g = \text{ht } I > 0$ and $\ell = \ell(I)$, and let J be a minimal reduction of I . Assume that I satisfies G_ℓ and $\text{depth } R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq \ell - g$. For every positive integer n write $L_n = L_n(J, I) = J^{n+1} : K_n(J, I)$. Then*

$$L_n = L_s$$

for every $n \geq s$, where s is as in Definition 3.1.

Proof. Let n be a fixed positive integer such that $n \geq s$. By Theorem 2.4 and by Lemma 3.3 there exists a positive integer t such that $K_n = \sum_{i=1}^t (J, b_i)^n$, where b_1, \dots, b_t are general elements in I , and $r_J((J, b_i)) \leq s$ for all $1 \leq i \leq t$. For simplicity we denote (J, b_i) by J_i and $K_n(J)$ by K_n since J is fixed. Notice that for all $1 \leq i \leq t$ we have that $J_i^{s+1} = JJ_i^s$ according to Lemma 3.2. Then

$$K_n = \sum_{i=1}^t J_i^n = \sum_{i=1}^t J^{n-s} J_i^s = J^{n-s} \sum_{i=1}^t J_i^s \subset J^{n-s} K_s.$$

In general $J^{n-s}K_s \subset K_n$ and thus

$$(3.2) \quad J^{n-s}K_s = K_n.$$

In conclusion

$$\begin{aligned} L_n = J^{n+1} : K_n &\stackrel{(3.2)}{=} J^{n+1} : J^{n-s}K_s \\ &= (J^{n+1} : J^{n-s}) : K_s \\ &\stackrel{(1)}{=} J^{s+1} : K_s = L_s, \end{aligned}$$

where (1) holds since the associated graded ring, $\text{gr}_J(R)$, of J is Cohen-Macaulay ([6, Theorem 9.1]) and $\text{ht } J > 0$ (c.f. [15, Remark 4.3]). \square

4. EXAMPLES

Finally we arrive at our goal. We are now ready to answer Question 1.4 with the next example using the results from the previous sections and the computer algebra program Macaulay 2 ([4]).

Example 4.1. Let $R = k[x, y, z]_{(x, y, z)} / (z^3)$, where k is an infinite field of characteristic 2. Consider the R -ideal $I = (x^2, y^2, xz, yz)$. Then

- (a) R is a 2-dimensional local Gorenstein ring with maximal ideal $\mathfrak{m} = (x, y, z)R$;
- (b) I is an \mathfrak{m} -primary ideal;
- (c) $g = \text{ht } I = 2$, $\ell = \ell(I) = 2$, $r = r(I) = 2$, and $r - \ell + g = 2$.

We claim that

$$J^{n+1} : I^n \subsetneq \text{core}(I) \subsetneq J^{n+1} : \sum_{b \in I} (J, b)^n$$

for any general minimal reduction J of I and any positive integer n .

The computation of $\text{core}(I)$ with Macaulay 2 ([4]) is done using general minimal reductions as in [2, Theorem 4.5]. That is, $\text{core}(I) = \bigcap_{i=1}^{\gamma(I)} J_i$, where $J_1, \dots, J_{\gamma(I)}$ are general minimal reductions. The sequence of ideals $\{J^{n+1} : I^n\}_{n \in \mathbb{N}}$ is a decreasing sequence and it stabilizes for $n \geq \max\{r_J(I) - \ell(I) + g, 0\} = 2$, according to [15, Corollary 2.3]. Also, $J^{n+1} : I^n \subset \text{core}(I)$ for $n \geq \max\{r_J(I) - \ell(I) + g, 0\} = 2$, according to Theorem 1.2. Hence it is enough to consider $J^{n+1} : I^n$ for $n \leq 2$. Using Macaulay 2 ([4]) it is easy to check that $\text{core}(I) \neq J^{n+1} : I^n$ for $n \leq 2$, where J is a general minimal reduction of I . Therefore

$$\text{core}(I) \neq J^{n+1} : I^n$$

for any general minimal reduction J of I and every positive integer n .

Notice that Theorem 2.4 provides an algorithm for computing the ideals K_n for any positive integer n . Once we obtain these ideals we can compute $L_n(J) = J^{n+1} : K_n(J)$. By Proposition 3.3 we have that $s = r_J((J, b))$, where b is a general element in I . In this case $s = 2$. By Theorem 3.4 we have that the sequence of the ideals $L_n(J)$ stabilizes after s steps. We then check that $\text{core}(I) \neq L_n(J)$ for $n \leq s = 2$ and therefore conclude that

$$\text{core}(I) \neq L_n(J)$$

for all positive integers n and any general minimal reduction J of I .

In order to see how close the $\text{core}(I)$ and the ideal $L_n(J)$ are we give a description in terms of generating sets obtained using Macaulay 2 ([4]). Note that the monomial ideal $J = (x^2, y^2)$ is a minimal reduction of I . Then $\text{core}(I) = (x^2z^2, y^2z^2, x^4, y^4, x^3yz, xy^3z, x^2y^2z, x^2y^3, x^3y^2)$ and $L_2(J) = (x^2y^2, y^2z^2, x^4, y^4, x^3yz, xy^3z, x^2y^2)$. Clearly $x^2y^2 \in L_2(J)$ and $x^2y^2 \notin \text{core}(I)$.

The question still remains: what is $\text{core}(I)$? According to [2, Theorem 4.5] in order to compute the core of an ideal I we only need to consider a finite intersection of general minimal reductions. Let $\gamma(I)$ be the number of reductions required in this intersection. Polini and Ulrich prove that the core is always contained in the ideals $L_n(J)$ for every n and any minimal reduction J of I ([15, Theorem 4.4]). On the other hand $L_n(J) \subset J$ for every n and every ideal J .

Combining these results we have

$$\text{core}(I) \subset \bigcap_{i=1}^{\gamma(I)} L_n(J_i) \subset \bigcap_{i=1}^{\gamma(I)} J_i = \text{core}(I),$$

where $J_1, \dots, J_{\gamma(I)}$ are general minimal reductions of I . Therefore

$$\text{core}(I) = \bigcap_{i=1}^{\gamma(I)} L_n(J_i),$$

where $J_1, \dots, J_{\gamma(I)}$ are general minimal reductions of I . In practice though it seems one can do much better. We test this in Example 4.1 using Macaulay 2 ([4]):

Example 4.2. In the case of Example 4.1 we have that for any positive integer $n \geq s = 2$

$$\text{core}(I) \neq L_n(J) \quad \text{but} \quad \text{core}(I) = \bigcap_{j=1}^2 L_n(J_j),$$

where J, J_1 , and J_2 are general minimal reductions of I , and s is as in Definition 3.1.

Remark 4.3. Notice that in the above example, $\ell = 2$ and $\gamma(I) \gg 0$. Using Macaulay 2 ([4]) we were able to construct a series of examples that support the following conjecture for the core of an ideal in arbitrary characteristic:

Conjecture 4.4. Let R be a local Gorenstein ring with infinite residue field, let I be an R -ideal with $g = \text{ht } I > 0$ and $\ell = \ell(I)$. Assume I satisfies G_ℓ and $\text{depth } R/I^j \geq \dim R/I - j + 1$ for $1 \leq j \leq \ell - g$. Let s be as in Definition 3.1. Then for any integer $n \geq s$

$$\text{core}(I) = \bigcap_{j=1}^{\ell} L_n(J_j),$$

where J_1, \dots, J_ℓ are general minimal reductions of I .

The above conjecture is consistent with the result of Polini and Ulrich in the case of analytic spread one (Theorem 1.3). Notice that Conjecture 4.4 implies that we need to intersect fewer of the ideals $L_n(J)$, since in general $\ell(I) \ll \gamma(I)$. The following examples support this conjecture.

Example 4.5. Let $R = k[x, y, z]_{(x, y, z)} / (z^5)$, where k is an infinite field of characteristic 2. Consider the R -ideal $I = (xy^2z^2 + y^5, xyz^3, x^4y + x^5, x^3yz)$. Then

- (a) R is a 2-dimensional local Gorenstein ring with maximal ideal $\mathfrak{m} = (x, y, z)R$;
- (b) I is an \mathfrak{m} -primary ideal;
- (c) $g = \text{ht } I = 2$, $\ell = \ell(I) = 2$, $r = r(I) = 4$, and $r - \ell + g = 4$.

Using the same methods as in Example 4.1 we compute $\text{core}(I)$ and the ideals $J^{n+1} : I^n$ and $L_n(J)$ for all $n \leq s = 4$, where s is as in Definition 3.1. We conclude that

$$J^{n+1} : I^n \subsetneq \text{core}(I) \subsetneq L_n(J),$$

for any general minimal reduction J of I and any positive integer n . Nevertheless for all $n \geq 4$

$$\text{core}(I) = \bigcap_{j=1}^2 L_n(J_j),$$

where J_1, J_2 are general minimal reductions of I . This is consistent with Conjecture 4.4.

Since $r - \ell + g = 4$ we may repeat the same computations with k an infinite field of characteristic 3. Using Macaulay 2 ([4]) we obtain $K_n = I^n$ for $n \geq 4$ and thus

$$\text{core}(I) = J^{n+1} : I^n = L_n(J),$$

for any minimal reduction J of I and $n \geq 4$. Notice that this does not contradict Conjecture 4.4.

In both Example 4.1 and Example 4.5 the analytic spread is 2. We now consider an example where the analytic spread is 3.

Example 4.6. Let $R = k[x, y, z, w]_{(x, y, z, w)} / (w^3)$, where k is an infinite field of characteristic 2. Consider the R -ideal $I = (x^5, x^2y^2w + z^5, xy^2w^2 + x^2z^2w, xyz^2w + y^5, y^2z^2w)$. Then

- (a) R is a 3-dimensional local Gorenstein ring with maximal ideal $\mathfrak{m} = (x, y, z, w)R$;
- (b) I is an \mathfrak{m} -primary ideal;
- (c) $g = \text{ht } I = 3$, $\ell = \ell(I) = 3$, $r = r(I) = 2$, and $r - \ell + g = 2$.

We again use the same methods as in Example 4.1 to compute $\text{core}(I)$ and the ideals $J^{n+1} : I^n$ and $L_n(J)$ for all $n \leq s = 2$, where s is as in Definition 3.1. We conclude that

$$J^{n+1} : I^n \subsetneq \text{core}(I) \subsetneq L_n(J)$$

for any general minimal reduction J of I and any positive integer n . Nevertheless for all $n \geq 2$

$$\text{core}(I) \neq \bigcap_{i=1}^2 L_n(J_i) \quad \text{but} \quad \text{core}(I) = \bigcap_{i=1}^3 L_n(J_i),$$

where J_1, J_2, J_3 are general minimal reductions of I . Thus this example provides yet more evidence for the truth of Conjecture 4.4.

In the case of analytic spread 4 we exhibit the following example. The computations become quite difficult for higher analytic spreads.

Example 4.7. Let $R = k[x, y, z, w, t]_{(x, y, z, w, t)} / (w^3)$, where k is an infinite field of characteristic 2. Consider the R -ideal $I = (x^5, x^2y^2w + z^5, xt^2w^2 + x^2z^2w, xyt^2w + y^5, yz^2wt, t^5)$. Then

- (a) R is a 4-dimensional local Gorenstein ring with maximal ideal $\mathfrak{m} = (x, y, z, w, t)R$;
- (b) I is an \mathfrak{m} -primary ideal;
- (c) $g = \text{ht } I = 4$, $\ell = \ell(I) = 4$, $r = r(I) = 2$, and $r - \ell + g = 2$.

Once again we use the same methods as in Example 4.1 to compute $\text{core}(I)$ and the ideals $J^{n+1} : I^n$ and $L_n(J)$ for all $n \leq s = 2$, where s is as in Definition 3.1. We conclude that

$$J^{n+1} : I^n \subsetneq \text{core}(I) \subsetneq L_n(J)$$

for any general minimal reduction J of I and any positive integer n . Nevertheless for all $n \geq 2$

$$\text{core}(I) \neq \bigcap_{i=1}^2 L_n(J_i), \quad \text{core}(I) \neq \bigcap_{i=1}^3 L_n(J_i), \quad \text{but} \quad \text{core}(I) = \bigcap_{i=1}^4 L_n(J_i),$$

where J_1, J_2, J_3, J_4 are general minimal reductions of I . Again the validity of Conjecture 4.4 is supported by this example.

In all the previous examples the rings that we considered were non-reduced. The next example is set in a regular local ring.

Example 4.8. ([5]) Let $R = k[x, y]_{(x, y)}$, where k is an infinite perfect field. Let $I = (x^2y^8, y^9, x^5y^3, x^4y^4, x^6)$. Then

- (a) R is a 2-dimensional regular local ring with maximal ideal $\mathfrak{m} = (x, y)R$;
- (b) I is an \mathfrak{m} -primary ideal;
- (c) $g = \text{ht } I = 2$, $\ell = \ell(I) = 2$, $r = r(I) = 2$, and $r - \ell + g = 2$;

Once again following the same ideas as before we conclude that for all $n \geq s = 2$

$$\text{core}(I) \neq L_n(J) \quad \text{and} \quad \text{core}(I) = \bigcap_{j=1}^2 L_n(J_j),$$

where J, J_1 , and J_2 are general minimal reductions of I and s is as in Definition 3.1.

5. ACKNOWLEDGMENTS

This paper is part of my Ph.D. thesis which was written under the supervision of Professor Bernd Ulrich at Purdue University. I would like to express my gratitude to Professor Bernd Ulrich for his guidance, advice, and constant support. I would also like to thank Professor Raymond C. Heitmann (University of Texas, Austin) for reading the manuscript and for his suggestions that improved the exposition of the paper.

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